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NONLINEAR BUBBLE OSCILLATIONS

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CALIFORNIA INSTITUTE OF TECHNOLOGY
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Abstract

Numerical solutions have been obtained for the motion of a gas bubble in an incompressible liquid when harmonic pressure oscillations are imposed. The mean pressure in the liquid was taken to be 1 atm and the ratio of oscillating pressure amplitude, A, to the mean pressure was given the values 0.2, 0.5, 1.0 and 2.0. The equilibrium bubble radius was chosen to be $R_o = 10^{-2}$, 10^{-3} and, 10^{-4} cm, and the angular frequency of the pressure variations was $\omega = 0.2 \times 10^5$, 0.7×10^5, and 1.2×10^5 per sec. The phenomena of bubble "explosion" or "collapse" were found for large A, i.e., 1.0 and 2.0. For the smaller values of A, the bubble radius varied with time within well-defined limits, but nonlinear effects were evident.

Introduction

The equation of motion for a gas bubble in a liquid acted upon by a variable pressure is nonlinear, and solutions for the radius as a function of time cannot be given explicitly in terms of simple, or familiar, functions. The example of a pressure variation harmonic in time is of particular interest since it corresponds to a sound field. In the treatment of the scattering of sound by a gas bubble it has been customary to linearize the equation of motion for the bubble radius. There are, however, many experimental situations in which the sound pressure amplitude is of the order of atmospheres. The acoustic linearization in the liquid remains valid with high precision but a question remains regarding the accuracy of the linearization of the bubble equation.

Numerical integrations of the bubble radius in harmonic pressure fields have been carried out by Noltingk and Neppiras[1]. The primary difference between their results and the present results lies in the greater range of parameters used here and also in the longer time over which the bubble histories are extended. There are more recent calculations by Borotnikova and Soloukin[2] but the published results are rather limited and more complete data is apparently not available in the general literature. Both their analysis and the previous work of Noltingk and Neppiras assume that the bubble oscillations are adiabatic. It is known, however, that the bubble oscillations in the range of concern are isothermal rather than adiabatic[3,4] and the calculations presented here are made on the isothermal basis. Calculations have also been made by Flynn[5] but his primary concern was with frequencies much higher than the sonic range which will be considered in this paper.

There are two physical effects which contribute elements of indefiniteness to any solutions for bubble motions. A bubble can of course be set into free oscillation; in the limit of very small pressure amplitudes such oscillations take place with the resonant frequency of the bubble. These free oscillations are excited by the initial conditions which are customarily used and which are used here. The bubble is taken to be in equilibrium and at rest under a static pressure P_o for $t < 0$; for $t > 0$ the oscillating pressure field $P_oA \sin\omega t$ is applied and subsequent motion of the bubble is determined. It may be shown explicitly from the linearized solution of the bubble motion that viscosity eventually damps out the resonance oscillations. A similar behavior is to be expected for the nonlinear solutions.

No consideration will be given here to possible physical mechanisms which give a stable bubble[6] or "nuclei" under the static pressure P_o for the indefinitely long period $t < 0$. The second physical effect which is not considered arises from rectified diffusion[7] which is the process by means of which a pulsating gas bubble grows in size by transport of dissolved gas from the liquid into the bubble. This rectified diffusion is, however, a slow process so that the solutions obtained will remain accurate for many cycles of the oscillating pressure field.

Formulation of the Problem

When the motions of a gas bubble are sufficiently slow so that the effects of compressibility may be neglected, the variations of the bubble radius with time are described by the equation[8,9]

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{p(R) - p_\infty}{\rho} \tag{1}$$

where ρ is the liquid density, $p(R)$ is the pressure in the liquid at the bubble wall, and p_∞ is the pressure in the liquid at a large distance from the bubble. It is assumed that effects which disturb spherical symmetry such as gravity may be neglected. The condition of stress continuity across the bubble boundary gives

$$p(R) = p_g(R) - \frac{2\sigma}{R} \quad . \qquad (2)$$

In Eq. (2) the surface tension constant is σ and $p_g(R)$ is the gas pressure in the bubble. The contribution of the vapor pressure of the liquid to the pressure within the bubble has been omitted. Since the bubble motions will be isothermal and since the bubble wall velocities are supposed to be small, the vapor pressure may be taken to be constant. A constant shift in the pressure level is not significant in the problem. The viscous stress, $-4\mu\dot{R}/R$, has been dropped from the right hand side of Eq. (2). This term has a significant effect on the bubble motion only after a long time, as has been remarked, and it eventually damps out the free oscillations.

For $t < 0$ the bubble is at rest with the equilibrium radius R_o so that

$$p_g(R_o) - \frac{2\sigma}{R_o} = P_o \qquad (3)$$

where P_o is the static pressure in the liquid. The equilibrium relation (3) determines the initial gas pressure $p_g(R_o)$ when P_o is fixed. The calculations which have been carried out in this study have all been made with $P_o = 1$ atm. For $t > 0$ the pressure at a distance from the bubble is supposed to be disturbed by the addition of a harmonic field so that the total applied pressure at a large distance from the bubble is

$$p_\infty(t) = P_o(1+A \sin\omega t) \quad . \tag{4}$$

The bubble motion is taken to have the initial conditions $R(0) = R_o$, $\dot{R}(0) = 0$. The gas pressure in the bubble at any radius is determined by the isothermal relation

$$p_g(R) = p_g(R_o)\left(\frac{R_o}{R}\right)^3 \tag{5}$$

Since the angular frequencies, ω, of the oscillating pressure will lie in the sonic range, the bubble will follow the isothermal law for all the radius values R_o of present concern. The problem is completely defined and the numerical integration can proceed in a straightforward way.

It is convenient to introduce the dimensionless radius

$$\eta = \frac{R}{R_o} \tag{6}$$

so that Eq. (1) becomes

$$\eta\ddot{\eta} + \frac{3}{2}\dot{\eta}^2 = c_1(\eta^{-3} - 1) + c_2(\eta^{-3} - \eta^{-1}) - c_1 A \sin\omega_o t \tag{7}$$

with

$$c_1 = \frac{P_o}{\rho R_o^2} \tag{8}$$

and

$$c_2 = \frac{2\sigma}{\rho R_o^3} \quad . \tag{9}$$

The initial conditions are

$$\eta(0) = 1 \tag{10}$$

$$\dot{\eta}(0) = 0 \quad . \tag{11}$$

Near $t = 0$, $\eta(t)$ will be very nearly unity so that for small time one may write

$$\eta(t) = 1 + \epsilon(t) \tag{12}$$

where $|\epsilon(t)| \ll 1$. Under this condition, the equation obtained from (7) for $\epsilon(t)$ may be linearized and integrated analytically. One finds readily that, for t near zero,

$$\eta(t) = 1 + \frac{c_1 A}{\left(\omega_r^2 - \omega_o^2\right)} \left(\frac{\omega_o}{\omega_r} \sin \omega_r t - \sin \omega_o t \right), \tag{13}$$

$$\dot{\eta}(t) = \frac{c_1 A \omega_o}{\left(\omega_r^2 - \omega_o^2\right)} \left(\cos \omega_r t - \cos \omega_o t\right), \tag{14}$$

where

$$\omega_r = \left(\frac{3P_o}{\rho R_o^2} \right)^{\frac{1}{2}} \left(1 + \frac{4\sigma}{3P_o R_o} \right)^{\frac{1}{2}} \tag{15}$$

is the isothermal resonant frequency. This initial solution (13) is used only up to a time τ at which $\epsilon(\tau)$ is still very small. The solution which is developed for $t > \tau$ is, of course, independent ot τ so long as τ is sufficiently small. This way of starting the numerical solution is for convenience only and serves to facilitate the development of the further integration.

Numerical solutions were obtained using an IBM 7094 Computer. The numerical integration of the equations was performed using a variable step size integration scheme. A small initial step size was given as an input; the integration scheme then proceeded to integrate each step with the largest step size possible, consistent with the specified accuracy. It was found that different choices of initial step size had no effect upon

the character of the solution.

Discussion of the Results

The results are presented in graphical form in Figs. 1 - 9. It may be observed that, as might be expected, the largest values of A, i.e., 1 and 2, lead to large variations in $\eta = R/R_o$ including the possibility of collapse or explosion. It is convenient to define a bubble "collapse" as a change in η by a factor of 10^{-2} or less; similarly a bubble "explosion" is defined as a change in η by a factor of 10^2 or more. As a general result it appears for large A that collapse occurs more frequently than explosion. It is of some interest to observe that, for the lower frequencies and for $R_o = 10^{-3}$ cm and 10^{-4} cm, no collapse occurs for A = 1. The rate of collapse which is found is clearly sufficiently great to lead to cavitation damage and cavitation noise. It also is evident that bubbles with radius values in the range 10^{-2} cm to 10^{-4} cm can act as nuclei for the cavitation phenomena usually observed. There is a trend for the higher frequencies to give, for all values of the initial radius and of A, a more violent radius change; there is a trend also at higher frequencies for collapse or explosion to occur earlier in the bubble history.

The bubble radius varies within well-defined limits for A = 0.5, and 0.2. In the figures which depict these solutions, comparisons with the linearized solutions are included; where they are omitted the two solutions essentially coincide. For the high frequencies the radius varies in a more erratic manner than for the low frequencies. There are also indications of more erratic variations in radius for the larger initial radii. The resonance oscillations which are excited have the effect of

giving a nonperiodic character to the solutions.

The results presented are for a phase of the applied oscillating pressure which gives $P_o A \sin \omega t$ with $A > 0$. Essentially the same behavior, as is shown here, is obtained for $A < 0$. For a given value of $|A|$, the general character of the solutions is preserved. This behavior is to be expected since the effect of explosion, or collapse, for example, appears only after a few cycles of the applied pressure. The specific phase at $t = 0$ would then be unimportant provided in all cases $\dot{R}(0)$ is always small.

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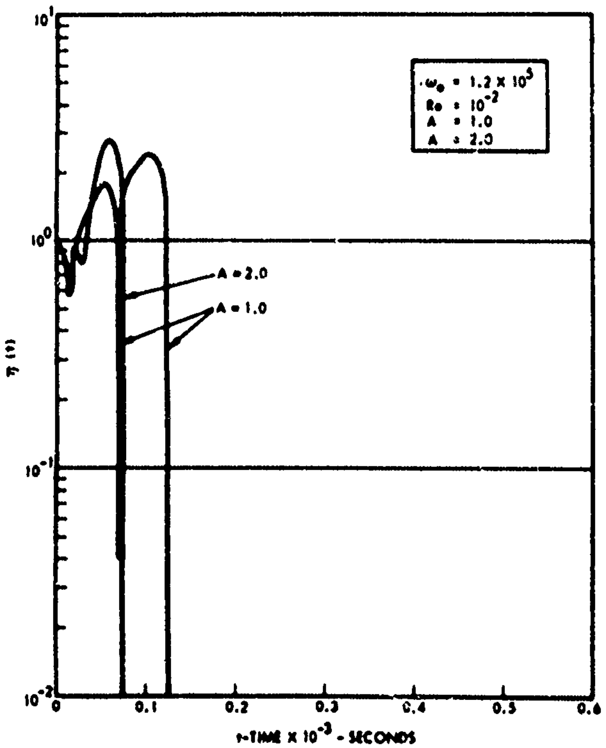
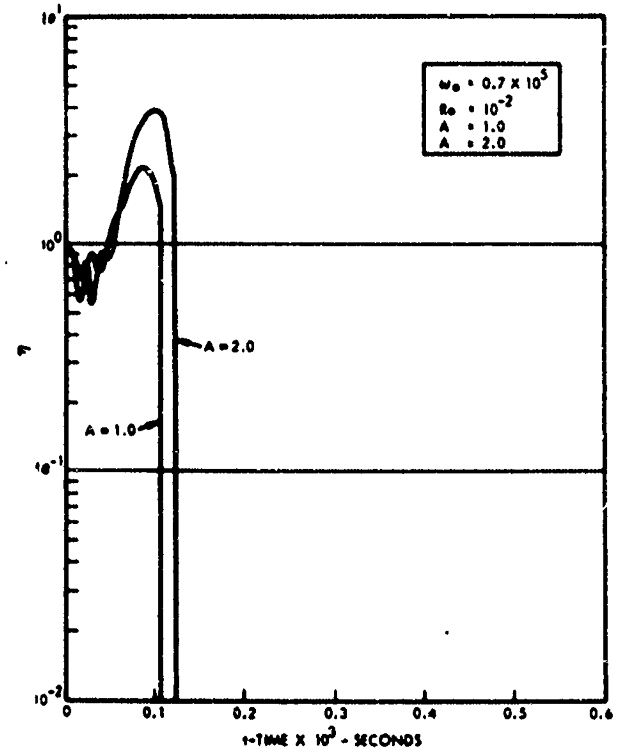


(a)



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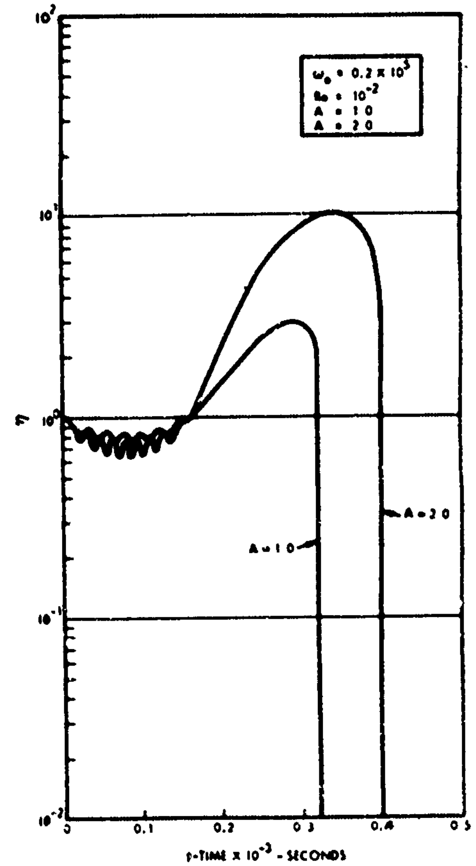


(c)

Figure 1. The ratio, $\eta(t)$, of the bubble radius to the initial radius, R_o, is shown as a function of time for $R_O = 10^{-2}$ cm. The mean pressure is 1 atm and A is the ratio of the oscillating pressure, applied at $t = 0$, to the mean pressure. ω, the angular frequency of the oscillating pressure, has the values 1.2×10^5/sec in (a), 0.7×10^5/sec in (b) and 0.2×10^5/sec in (c).

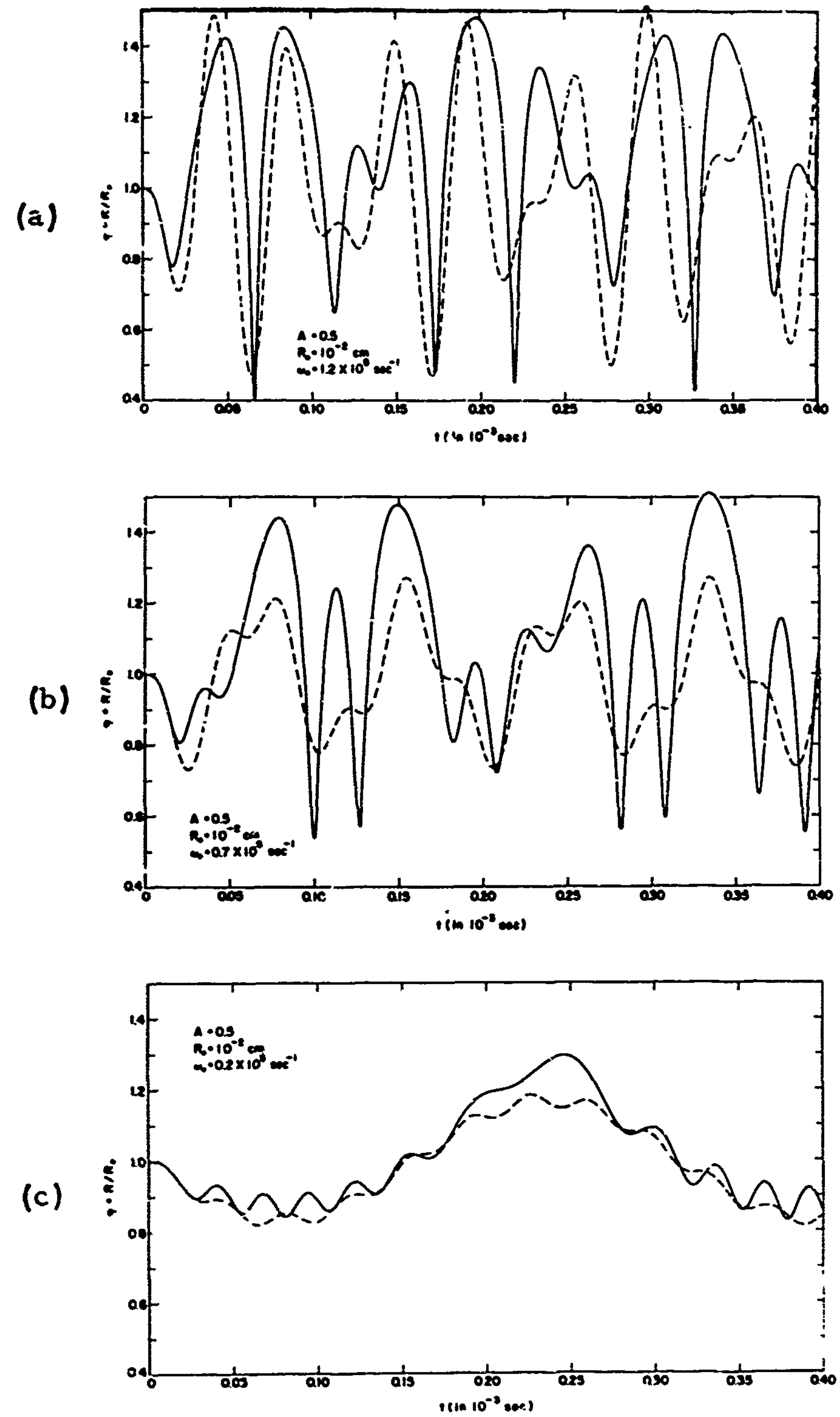
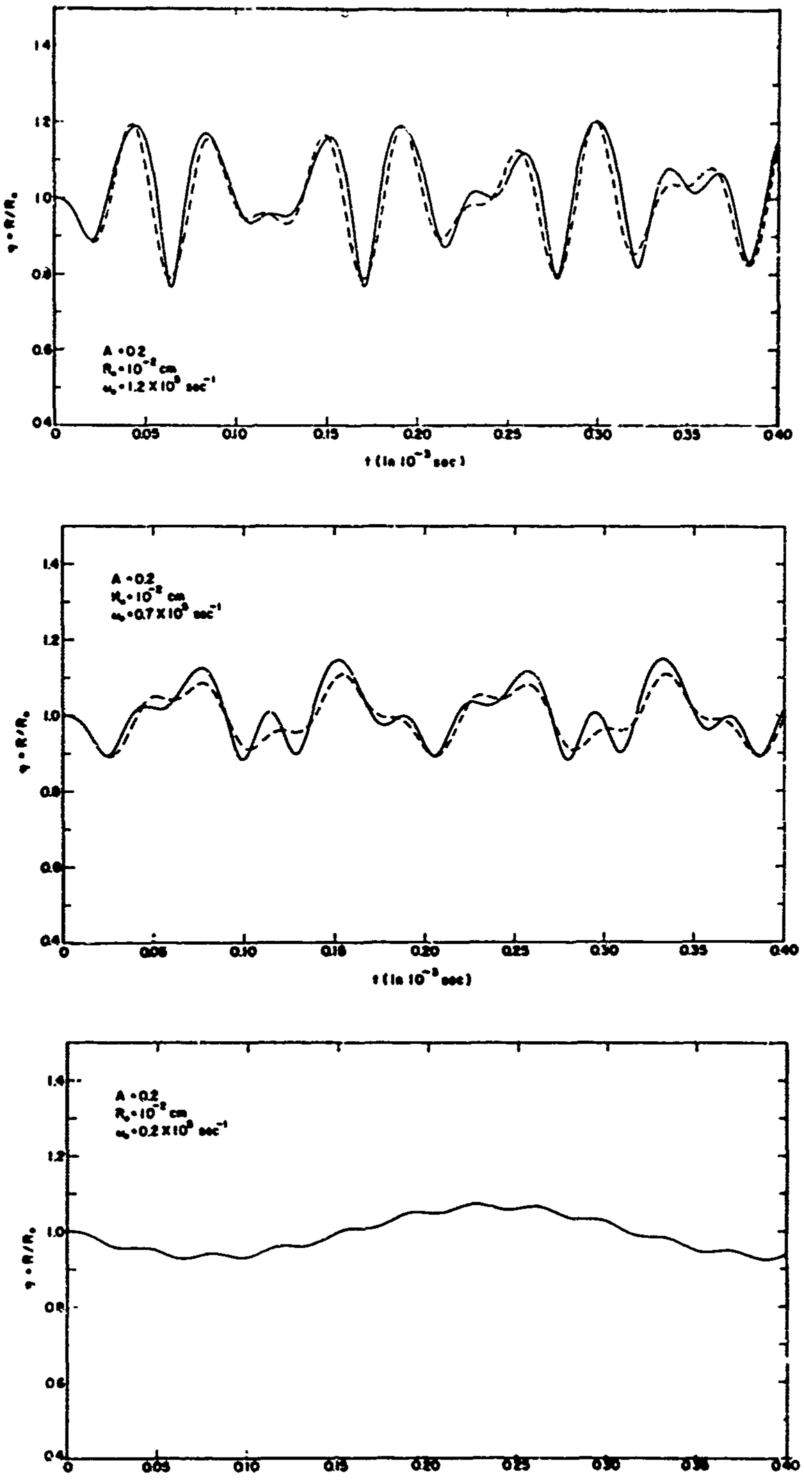


Figure 2. The ratio, $\eta(t)$, of the bubble radius to the initial radius, R_o, is shown as a function of time for $R_o = 10^{-2}$ cm. The mean pressure is 1 atm and A is 0.5. ω, the angular frequency of the oscillating pressure is 1.2×10^5/sec in (a), 0.7×10^5 sec in (b), and 0.2×10^5/sec in (c). The dashed curves show the linearized solutions.



(a)

(b)

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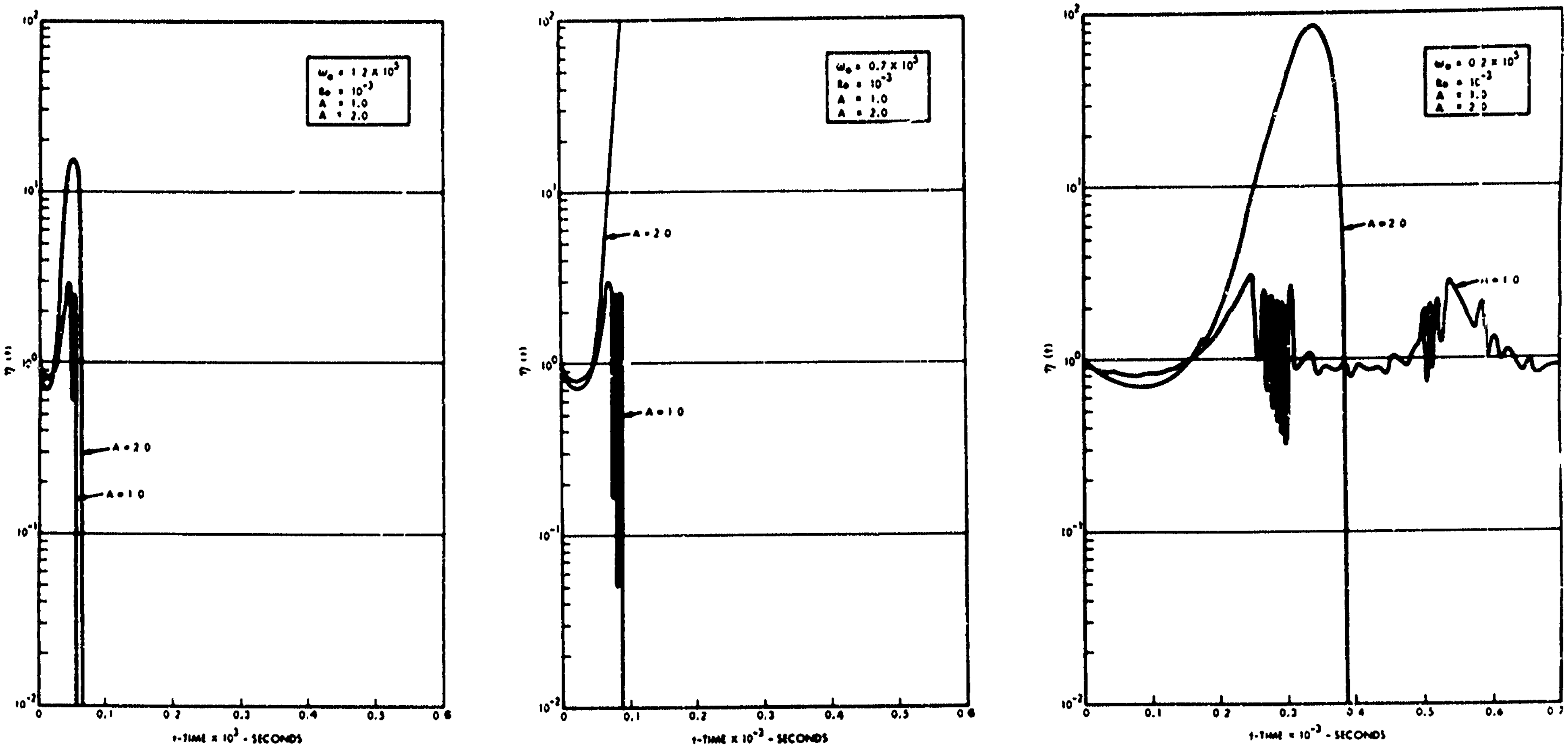
Figure 3. The ratio, $\eta(t)$, of the bubble radius to the inidial radius, R_o, is shown as a function of time for $R_o = 10^{-2}$ cm. The mean pressure is 1 atm and A is 0.2. ω, the angular frequency of the oscillating pressure is 1.2×10^5/sec in (a), 0.7×10^5/sec in (b), and 0.2×10^5/sec in (c). The dashed curves show the linearized solutions.



(a) (b) (c)

Figure 4. The ratio, $\eta(t)$, of the bubble radius to the initial radius, R_o, is shown as a function of time for $R_o = 10^{-3}$ cm. The mean pressure is 1 atm and A is the ratio of the oscillating pressure applied at $t = 0$, to the mean pressure. ω, the angular frequency of the oscillating pressure is 1.2×10^5 /sec in (a), 0.7×10^5/sec in (b), 0.2×10^5/sec in (c).

(a) (b) (c)

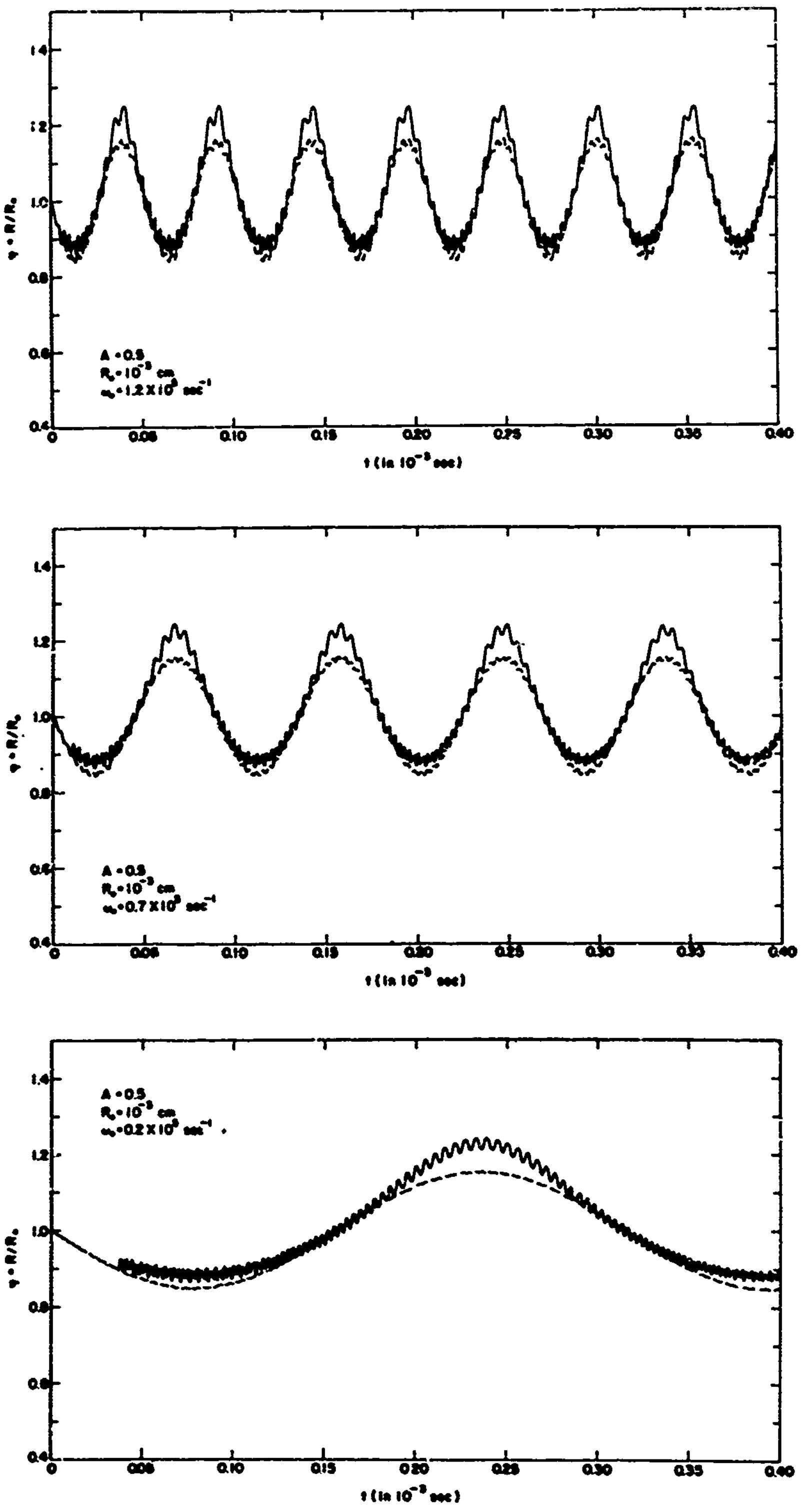


Figure 5. The ratio, $\eta(t)$, of the bubble radius to the initial radius, R_o, is shown as a function of time for $R_o = 10^{-3}$ cm. The mean pressure is 1 atm and A is 0.5. ω, the angular frequency of the oscillating pressure is 1.2×10^5/sec in (a), 0.7×10^5/sec in (b), and 0.2×10^5/sec in (c). The dashed curves show the linearized solutions.

(a)

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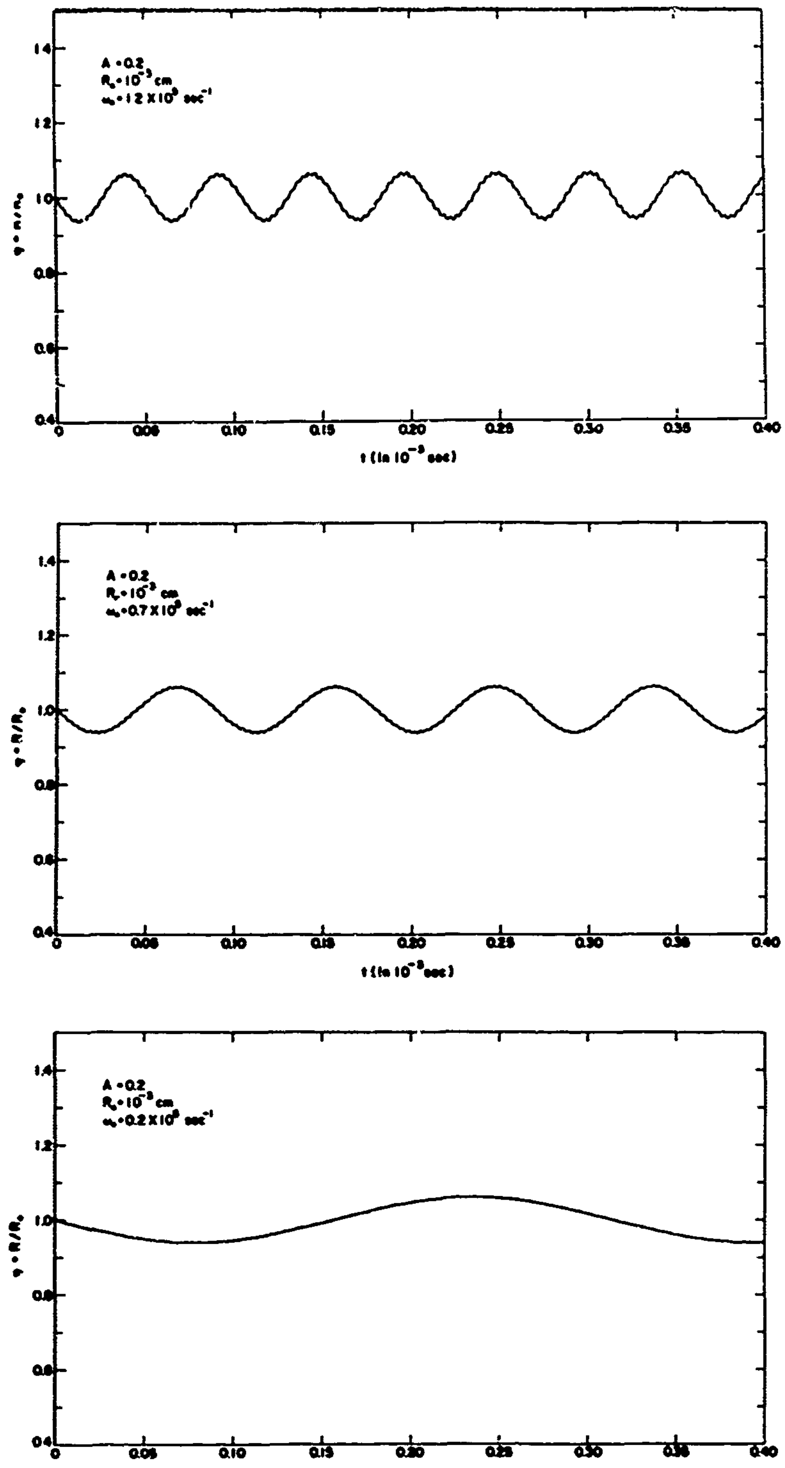
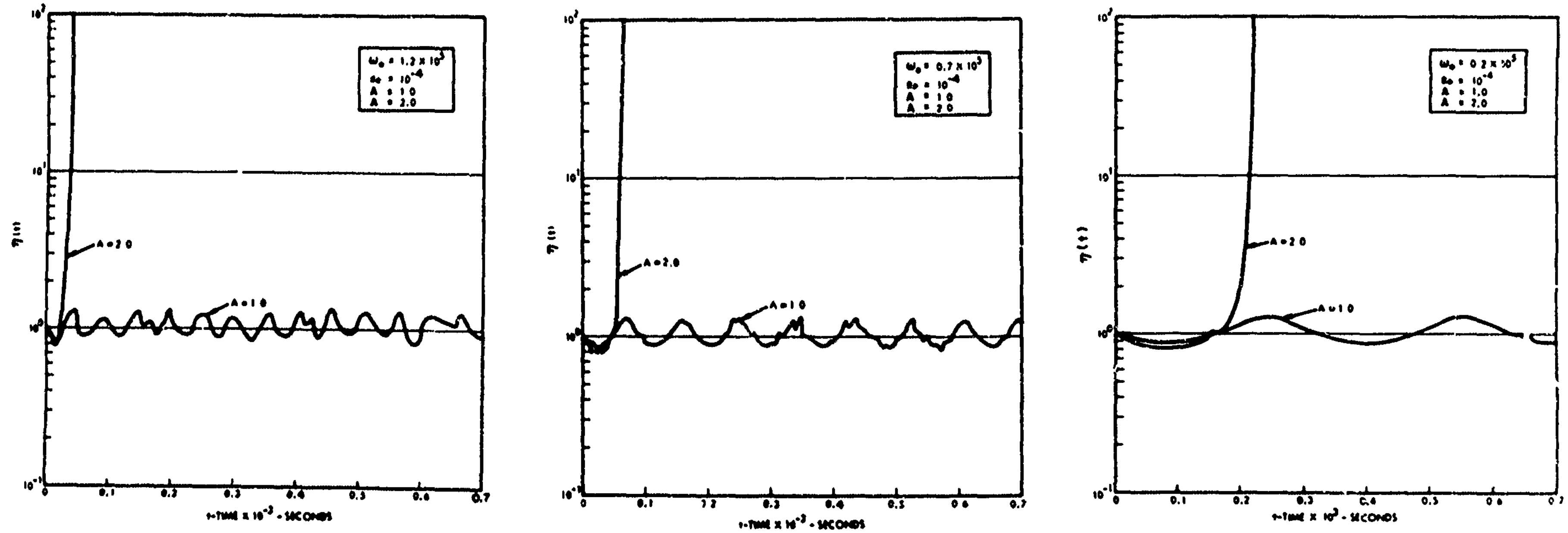


Figure 6. The ratio, $\eta(t)$, of the bubble radius to the initial radius, R_o, is shown as a function of time for $R_o = 10^{-3}$ cm. The mean pressure is 1 atm and A is 0.2. ω, the angular frequency of the oscillating pressure is 1.2×10^5/sec in (a), 0.7×10^5/sec in (b), and 0.2×10^5/sec in (c).



(a) (b) (c)

Figure 7. The ratio, $\eta(t)$, of the bubble radius to the initial radius, R_o, is shown as a function of time for $R_o = 10^{-4}$ cm. The mean pressure is 1 atm and A is the ratio of the oscillating pressure applied at $t = 0$, to the mean pressure. ω, the angular frequency of the oscillating pressure is 1.2×10^5/sec in (a), 0.7×10^5/sec in (b), and 0.2×10^5/sec in (c).

(a)

(b)

(c)

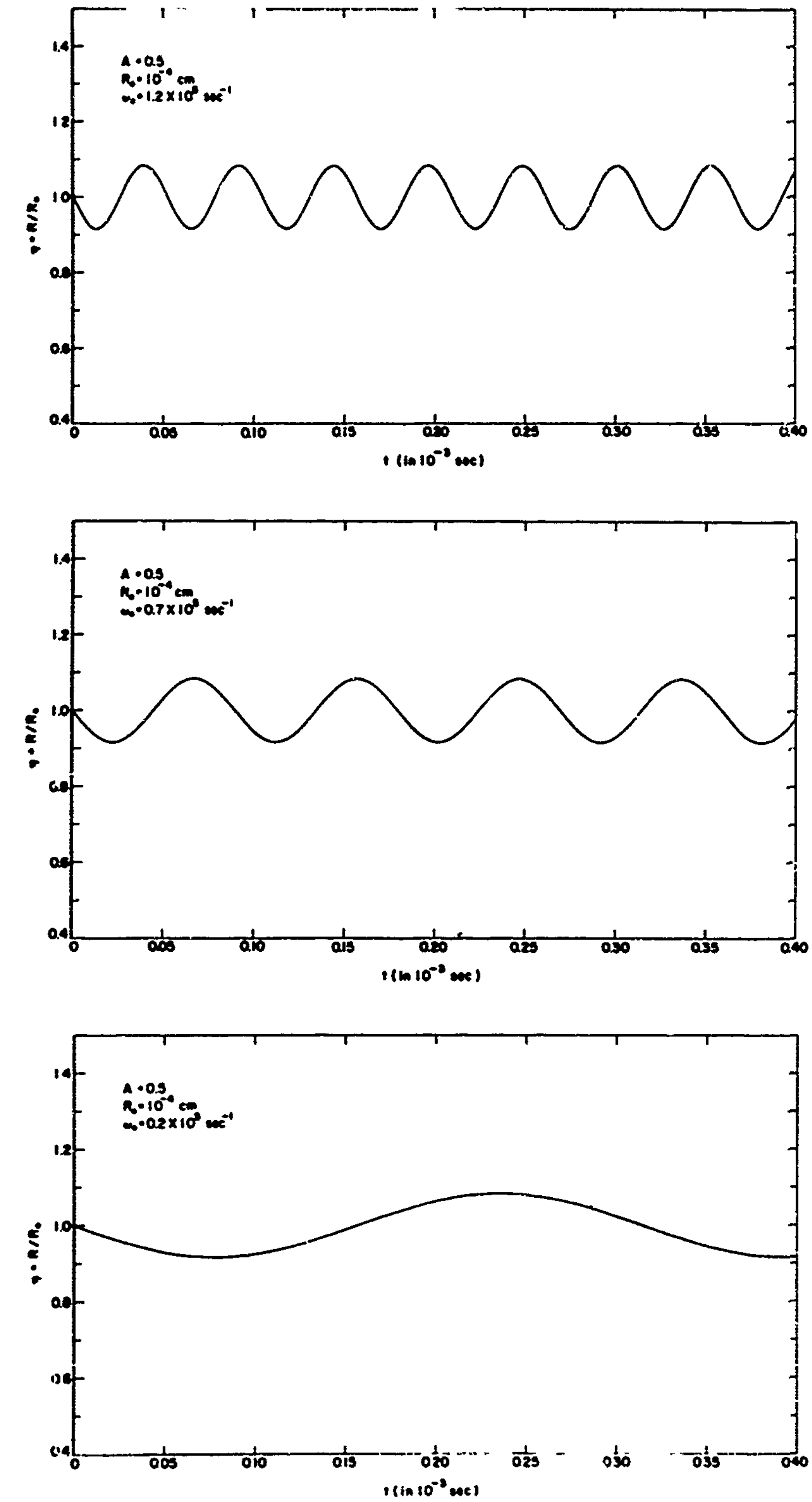


Figure 8. The ratio, $\eta(t)$, of the bubble radius to the initial radius, R_o, is shown as a function of time for $R_o = 10^{-4}$ cm. The mean pressure is 1 atm and A is 0.5. ω, the angular frequency of the oscillating pressure is 1.2×10^5/sec in (a), 0.7×10^5/sec in (b), and 0.2×10^5 /sec in (c).

(a) (b) (c)

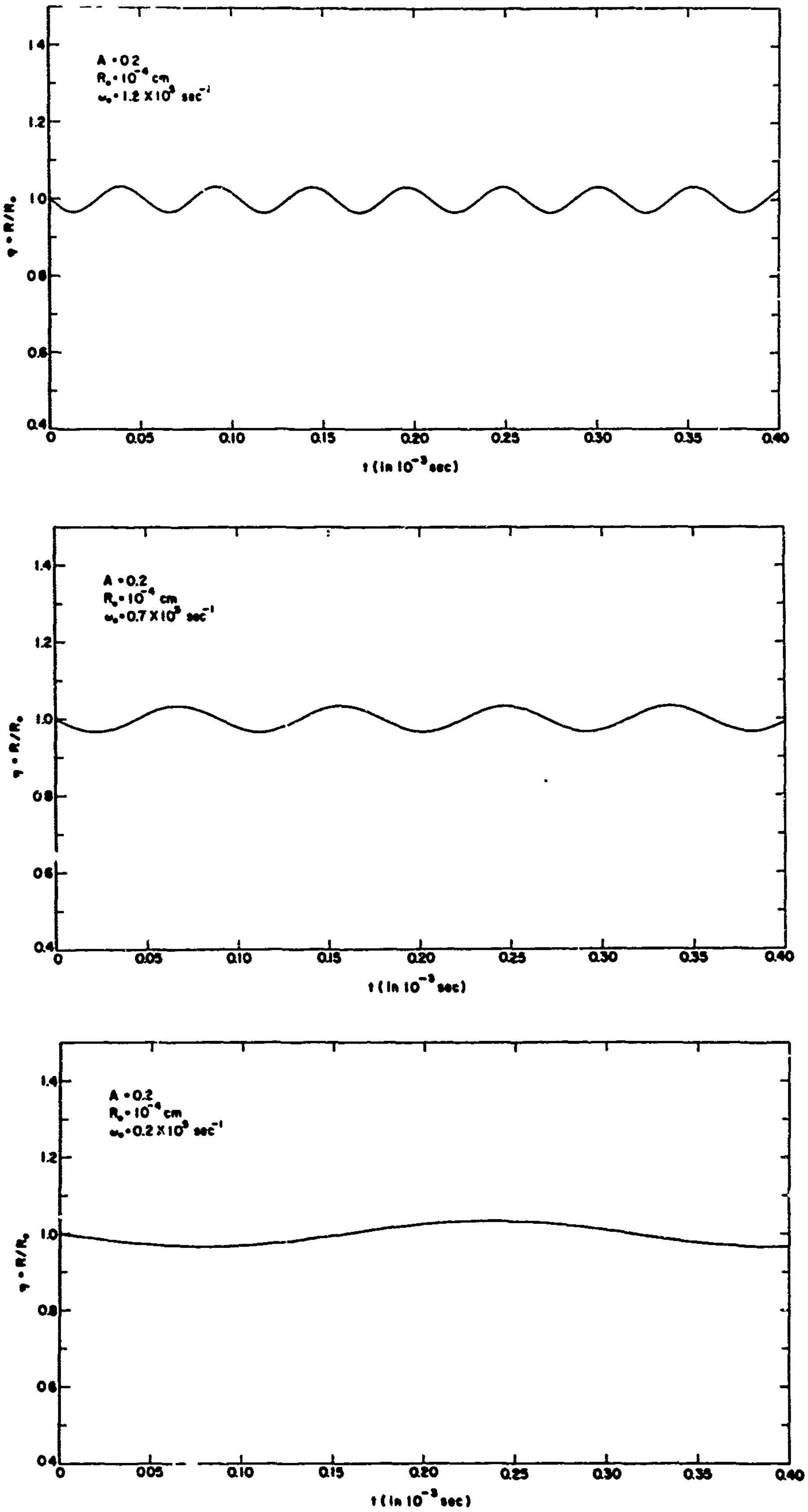


Figure 9. The ratio, $\eta(t)$, of the bubble radius to the initial radius, R_o, is shown as a function of time for $R_o = 10^{-4}$ cm. The mean pressure is 1 atm and A is 0.2. ω, the angular frequency of the oscillating pressure is 1.2×10^5/sec in (a), 0.7×10^5/sec in (b), and 0.2×10^5/sec in (c).

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